NASA CONTRACTOR REPORT

NASA CR-440

LAMINAR BOUNDARY LAYER ON A CONE IN SUPERSONIC FLOW WITH UNIFORM MASS TRANSFER

by Paul A. Libby

Prepared under Grant No. NGR-05-009-025 *by*
UNIVERSITY OF CALIFORNIA
San Diego, Calif.
for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • APRIL 1966

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SUMMARY

A solution for the laminar boundary layer on a cone with uniform mass transfer is obtained. The velocity field is found for either suction or injection but the related solution for the energy field is subject to an energy balance at the exposed surface and is therefore valid only for injection. This latter solution is equally applicable to certain species fields as well. The present results along with those presented previously for the two-dimensional case permit a comparison of the effect of injection on boundary layers over two-dimensional and conical surfaces.

INTRODUCTION

The characteristics of laminar boundary layers involving mass transfer are of interest in connection with the thermal protection of surfaces subjected to high-energy gas flows and with boundary layer control. The essential simplification resulting from the assumption of flow similarity has led to its widespread application to the theoretical treatment of such boundary layers. Indeed, in certain applied problems, e.g., at stagnation points and on surfaces involving steady-state sublimation, the similarity assumption is exactly applicable. Much less attention has been devoted to nonsimilar boundary layers with mass transfer although they arise in problems of practical interest; e.g., experimental research of a fundamental nature is often carried out on porous surfaces which lead to uniform rates of suction or injection and thus to nonsimilar flows. The present report deals with the boundary layer on a cone in supersonic flow with uniform mass transfer.

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*This report is an extended version of a paper accepted for publication in the Physics of Fluids. In the abbreviated version the previous analysis of reference 1 is relied on heavily for notation and details; the present extended version is more complete and self-contained.

The analysis develops first the velocity field in the boundary layer on the cone; this will be found to be applicable to suction as well as to injection. However, the next stage of the analysis pertains to the energy and species conservation fields associated with this velocity distribution under steady flow conditions; thus energy and species balance conditions are applied at the porous surface and the resultant solutions are applicable only to the case of injection. This is generally the case of greater applied interest.

The present analysis should be considered a generalization to the case of a cone of previous two-dimensional solutions with uniform mass transfer. In the context of modern transformations of the equations describing laminar boundary layers for both two-dimensional and axisymmetric flows, it may be a priori surprising that these two cases must be treated separately. However, the requirement of uniform mass transfer leads mathematically to two distinct cases and the analysis for one cannot be interpreted so as to apply to the other. Physically, the distinct behavior in the two cases appears to be related to the different effect of the uniform mass transfer on the boundary layer growth which in both cases is nonparabolic, a symptom of nonsimilarity.

The problem of the two-dimensional boundary layer with uniform mass transfer has been considered by several authors; Iglish (reference 2) presents a solution for incompressible flow with uniform suction. Lew and Fannucci (reference 3) provide the extension to compressible flows with either uniform suction or injection. The analyses of both references 2 and 3 deal primarily with the velocity field and involve the numerical solution of a partial differential equation which is free from parameters and which thus has been integrated once-for-all.

Libby and Chen (reference 4) recently treated the two-dimensional case by carrying out an expansion in terms of a mass transfer parameter. This permits the velocity, energy and species fields, the latter two involving balance conditions at the porous surface, to be determined from the successive solution of ordinary differential equations. The velocity field provided by reference 4 duplicates that obtained previously in references 2 and 3 but its availability and describing form permit the energy and species fields, which had previously been largely ignored, to be readily obtained. Of interest in these solutions is the distribution of wall enthalpy and wall concentration of the various species present. For example, at the leading edge of the plate the enthalpy at the porous surface is equal to the stagnation enthalpy of the external flow; downstream from the leading edge the wall enthalpy approaches that in the coolant chamber. Similar considerations apply to the composition at the porous surface. Such streamwise behavior is, of course, symptomatic of nonsimilar flows. No generalization of the aforementioned two-dimensional analyses to the axisymmetric case appears to have been performed. However, Libby (reference 4) in connection with an analysis of mass transfer effects treated as perturbations recently performed a calculation related to that presented here but involving only the first-order effect of injection. The principal motivation and utility of the analysis of reference 4 are the possibility of computing the effects of arbitrarily distributed mass transfer.

It is perhaps of interest to note that none of the aforementioned analyses are able to answer the question of whether a constant, finite injection rate, $(\rho v)_w$, applied over a finite streamwise length, x, leads to "blow-off," i. e., to a point of zero skin-friction. The earlier studies of boundary layers with mass transfer based on momentum integral methods indicated a negative answer to this question but Stewartson (reference 5) has recently called attention to some unpublished results which imply an affirmative answer.*
Studies of the behavior of boundary layers in the neighborhood and downstream of the "blow-off" point are of considerable interest but are outside the scope of the present paper.

The present analysis follows closely that of reference 1 and provides its extension to the axisymmetric case; the solution for the velocity field is presented first and then applied to the determination of the energy and species fields.

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*In terms of the mass transfer parameter of reference 1, namely $\epsilon \equiv [-(\rho v)_w/\rho_e \mu_e u_e](s/2)^{1/2}$, "blow-off" occurs at $\epsilon = -0.606$.

SYMBOLS

c_f	local skin-friction coefficient
f	transformed stream function, cf. equation (2)
G_n	unit solutions for the energy and species fields, cf. equation (20)
g	ratio of stagnation enthalpies, $h_s/h_{s,e}$
h	static enthalpy
h_s	stagnation enthalpy, $(u^2/2) + h$
N_n	unit solutions for the velocity field, cf. equation (11)
$R_n^{(1)}, R_n^{(2)}$	inhomogeneous terms, cf. equations (12) and (21)
r	cylindrical coordinate
s	transformed streamwise coordinate, cf. equation (1)
u	streamwise velocity component
v	normal velocity component
x	streamwise coordinate along the surface
Y_i	mass fraction of species i
y	coordinate normal to the surface
α	cone parameter, $\alpha \equiv \sin \theta_c$
ϵ	mass transfer parameter for two-dimensional case
η	transformed normal coordinate, cf. equation (1)
θ_c	cone half-angle
μ	viscosity coefficient
ρ	mass density
χ	transformed streamwise coordinate, cf. equation (8).

<u>Subscripts</u>

c	refers to conditions in the coolant chamber
e	refers to conditions in the external stream
w	refers to conditions at the surface, $y \propto \eta = 0$
s, e	refers to stagnation conditions in the external stream.

THE VELOCITY FIELD

The geometrical and dynamical variables being considered are shown schematically in figure 1. The velocity distribution throughout the boundary layer may be obtained in terms of transformed variables from the x-wise momentum and the mass conservation equations alone provided there is employed the common assumption that the ratio $\rho\mu/\rho_e\mu_e \simeq 1$. The transformed variables are the well-known Levy-Lees η, s (cf., e.g., references 5 and 6) defined for the present problem according to

$$\eta \equiv \rho_e u_e r (2s)^{-1/2} \int_0^y (\rho/\rho_e) \, dy'$$
$$s \equiv \rho_e \mu_e u_e \int_0^y r^2 \, dx = \rho_e \mu_e u_e \alpha^2 (x^3/3) \tag{1}$$

where $r = x \sin \theta_c = \alpha x$. The velocity components are found from the transformed stream function $f(s, \eta)$ by means of the equations

$$u = u_e f_\eta$$
$$v = - [(2s)^{1/2}/\rho r]\{\rho_e \mu_e u_e r^2 [(f/2s) + f_s] + \eta_x f_\eta\} \tag{2}$$

A useful result from the second of equations (2) relates the distributions of $f(s, 0) \equiv f_w(s)$ and of $(\rho v)_w$, i.e.,

$$\frac{d}{ds} [(2s)^{1/2} f_w] = - \frac{(\rho v)_w}{\rho_e \mu_e u_e r} \tag{3}$$

Finally, the partial differential equation for $f(s, \eta)$ is (cf., e.g., reference 6)

$$f_{\eta\eta\eta} + f f_{\eta\eta} - 2s (f_\eta f_{s\eta} - f_s f_{\eta\eta}) = 0 \tag{4}$$

Consider next the conditions to which the solution of equation (4) is to be subject; if it is assumed that $s f_s \to 0$ for all η as $s \to 0$ and that $f(0, 0) = f_w(0) = 0$ as will be shown to be the case below, indeed $f_w \propto s^{1/6}$, then

$$f_\eta(0, \eta) = f_0'(\eta) \tag{5}$$

where f_0 is the Blasius function and where the prime denotes differentiation with respect to η. This behavior near the origin of the boundary layer should be contrasted with that which prevails for flows with $f_w(0) \neq 0$; this is the case for the boundary layer on a cone in supersonic flow if $(\rho v)_w \propto s^{-1/2}$ at least as $s \to 0$. Then the effect of mass transfer prevails even at $s = 0$. In the present case the uniform mass transfer must persist over some distance from the origin before it influences the boundary layer.

The conditions on $f_\eta(s, \eta)$ far from and at the porous surface are the usual

$$f_\eta(s, \infty) = 1 \quad , \quad f_\eta(s, 0) = 0 \quad . \tag{6}$$

The final condition introduces the effect of mass transfer; from equation (3) it is found that for uniform mass transfer, i. e., $(\rho v)_w$ = constant,

$$f(s, 0) = f_w(s) = \{[-(\rho v)_w/\rho_e u_e] (\rho_e u_e)^{1/2} \mu_e^{-2/3} (3^{2/3}/\alpha^{1/2} 2^{3/2})\} s^{1/6} \quad . \tag{7}$$

Thus $f_w(s) \propto s^{1/6}$ as contrasted to the two-dimensional case in which $f_w(s) \propto s^{1/2}$ and to the case of similar flow in which $f_w(s)$ = constant. For a given uniform, injection rate, given cone angle and given properties of the external flow the factor within { } in equation (7) is known.

It is convenient to introduce a new, nondimensional, independent variable, χ, such that the solution for f is considered as $f(\chi, \eta)$ where

$$\chi \equiv \{ \quad \} s^{1/6} \quad . \tag{8}$$

Note that $\chi < 0$ for injection and $\chi > 0$ for suction. Note further that χ corresponds to $\epsilon(s)$ used in reference 1 for the two-dimensional case; this correspondence will be of interest in some considerations below of the relative effect of mass transfer on the boundary layer on wedges and cones.

Equations (4) through (7) then yield the following mathematical problem:

$$f_{\eta\eta\eta} + f\, f_{\eta\eta} - (\chi/3)\,(f_{\eta}\, f_{\eta\chi} - f_{\chi}\, f_{\eta\eta}) = 0$$
$$f_{\eta}(0,\eta) = f_0'(\eta)\ ;\quad f_{\eta}(\chi,\infty) = 1\ ;\quad f_{\eta}(\chi,0) = 0\ ;\quad f(\chi,0) = \chi\ . \tag{9}$$

The quantity of most technological interest from the solution for the velocity field is the skin-friction coefficient; in terms of $f(\chi,\eta)$ this is

$$c_f = (6)^{1/2}\,(\rho_e u_e x/\mu_e)^{-1/2}\, f_{\eta\eta}(\chi,0) \tag{10}$$

so that the distribution of wall values of $f_{\eta\eta}$ is of interest. Note, of course, that in equation (10) $\chi = \chi(x)$.

The solution of equations (9) can be carried out by finite difference procedures for both positive and negative values of χ starting at $\chi = 0$. Such a solution would correspond to the extension to the cone case of the two-dimensional calculations of Iglish (reference 2) for $\chi > 0$ and of Lew and Fannucci (reference 3) for $\chi \lesseqgtr 0$. A less ambitious computing task is involved if a series solution analogous to that of reference 1 is found; such a solution appears to be adequate for many applied problems. Thus assume

$$f(\chi,\eta) = f_0(\eta) + \sum_{n=1}^{\infty} \chi^n N_n(\eta)\ . \tag{11}$$

Substitution into equation (9) and collection of like-powers in χ leads to the infinite array of equations

$$\begin{aligned} N_n''' + f_0 N_n'' - (n/3) f_0' N_n' + [1 + (n/3)] f_0'' N_n &= 0\ , \qquad n = 1 \\ &= R_n^{(1)}\ , \qquad n \geq 2 \end{aligned} \tag{12}$$

where the $R_n^{(1)}$ are nonlinear, inhomogeneous terms depending on $N_k(\eta)$, $k = 1, 2, \ldots, n-1$. For example,

$$R_2^{(1)} = (1/3)\left[(N_1')^2 \cdot N_1 N_1''\right]$$
$$R_3^{(1)} = N_1' N_2' - (1/3) N_1 N_2'' - (2/3) N_2 N_1'' \quad . \tag{13}$$

The solutions of equations (12) are subject to the boundary conditions

$$\begin{aligned} N_n'(0) &= N_n'(\infty) = 0 \\ N_n(0) &= 1 \quad , \qquad n = 1 \\ &= 0 \quad , \qquad n \geq 2 \end{aligned} \tag{14}$$

The first five functions $N_n(\eta)$ have been found numerically. The crucial values for generating these functions in detail are $N_n''(0)$; accordingly, these have been listed in Table I. In addition the "velocity profiles" are usually of most graphic interest so these have been given in figure 2. Also of interest is the variation of $(f_{\eta\eta})_w$ with χ; this is shown in figure 3 for four and five terms in the series. From a practical viewpoint the availability of only five terms would appear from figure 3 to be unimportant provided $|\chi| \lesssim 0.6$.

The availability of this solution for the velocity field in the axisymmetric case along with the previously available solutions for the two-dimensional case permits a comparison of the relative effects of mass transfer on cones and wedges be carried out. For this purpose it is of interest to compare $\epsilon(s)$ and $\chi(x)$; thus consider a wedge and a cone under flow conditions so that ρ_e, μ_e, u_e are the same in the two cases and let $(\rho v)_w/\rho_e u_e$ be common. Then at equal distances from the leading edge and from the apex, i.e., at the same station x, it can be shown that

$$\epsilon = (2/3^{1/2})\,\chi \quad . \tag{15}$$

Moreover, the ratio of the skin-friction coefficients at the same x-station becomes

$$\frac{(c_f)_{2-D}}{(c_f)_{cone}} = \frac{f_{\eta\eta}(s, 0)}{3^{1/2}\, f_{\eta\eta}(\chi, 0)} \tag{16}$$

where it is to be understood that the s and χ in the arguments of $f_{\eta\eta}$ are related according to equation (15) in order to assure comparison at the same x-station. Note that if $f_{\eta\eta}(s, 0) = f_{\eta\eta}(\chi, 0)$, then equation (16) is simply the well-known relation between the skin-friction on a cone and wedge without mass transfer. If $f_{\eta\eta}(s, 0) = f_{\eta\eta}(\chi, 0)$ with mass transfer, then on the basis of the comparison as carried out here, mass transfer is considered equally effective in both geometries.

In figure 4 the ratio of the two skin-friction coefficients is given for a range of χ. The wide excursion of this ratio from the zero mass transfer value of 0.578 will be noted. From this figure it will be seen that the effect of mass transfer in altering the skin friction is greater on wedges than on cones when the comparison is carried out on the basis described above. Note, of course, that the total mass added through the porous surface from the apex of the cone to the generic station x per unit length of perimeter at x is one-half that added from the leading edge of the wedge over the same x-wise length per unit length parallel to the leading edge.

THE ENERGY FIELD

The energy field will be described in terms of the ratio of stagnation enthalpies, $g \equiv h_s/h_{s,e}$. In addition to the previously employed assumption, $\rho\mu/\rho_e\mu_e \simeq 1$, it will be assumed here, as is frequently done, that the Prandtl number is unity, that a single diffusion coefficient exists and that the Schmidt number based thereon is unity. Then the energy equation in χ, η variables is (cf., e.g., reference 6)

$$g_{\eta\eta} + f\, g_\eta - (\chi/3)(f_\eta\, g_\chi - f_\chi\, g_\eta) = 0 \tag{17}$$

where it is assumed that $f(\chi, \eta)$ is known from the above analysis. The solutions to equation (17) apply to a variety of physical cases depending on the initial and boundary conditions at the porous surface. Here there will be considered only the case of injection $(\chi < 0)$; moreover, it will be assumed that the convective heat transfer from the gas to the porous surface is absorbed by the injected gas in passing from an injection chamber, where its enthalpy ratio is constant and denoted by g_c, to the exposed surface where its enthalpy ratio is variable and denoted by $g_w = g_w(\chi)$, to be determined. Thus under steady flow conditions the convective heat transfer is found to be related to the enthalpy difference $(g_w - g_c)$ according to*

$$(g_\eta)_w = (4/3)(-\chi)(g_w - g_c) \quad . \tag{18}$$

Note that both the case of a coolant being injected to provide thermal insulation, i. e., $g_c < 1$, and the case of a heated gas being injected to heat the surface and the boundary layer, i. e., $g_c > 1$, may be treated by the present analysis; perhaps the former case is of greater current interest. It is perhaps worth noting that if in addition to the convective heat load there exists a uniform additional thermal load, e. g., due to radiation, then this heat balance condition still prevails but with a redefined g_c which accounts for this added load.

From equation (18) and quite separately on physical grounds it is expected that at the origin $(\chi = 0)$, $g_w(0) = 1$ so that the initial and external stream conditions are

$$g(\chi, \infty) = g(0, \eta) = 1 \quad . \tag{19}$$

The solution of the problem posed by equations (17) through (19) again may be found by finite difference calculations; here a series solution is found in the form

$$g(\chi, \eta) = 1 + (g_c - 1) \sum_{n=1}^{\infty} \chi^n G_n(\eta) \tag{20}$$

where the $G_n(\eta)$ functions are given by an array of ordinary differential equations

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*Reference 1 provides a detailed derivation of the energy and species balance conditions; little change is necessary to obtain the conditions for the cone case.

$$G_n'' + f_0 G_n' - (n/3)\, f_0' G_n = 0 \;, \qquad n = 1$$

$$= R_n^{(2)} \;, \qquad n \geq 2 \qquad (21)$$

and are subject to the conditions

$$G_n'(0) = -(4/3)\, G_{n-1}(0) \;, \qquad n \geq 2$$

$$= (4/3) \;, \qquad n = 1 \;.$$

Again the $R_n^{(2)}$ functions are known, nonlinear functions of the previous N_n and G_n functions; for example

$$R_2^{(2)} = -(4/3)\, N_1\, G_1' - (1/3)\, N_1'\, G_1$$

$$R_3^{(2)} = -(4/3)\, N_1\, G_2' + (2/3)\, N_1'\, G_2 + (1/3)\, N_2'\, G_1 - (5/3)\, N_2\, G_1' \;. \qquad (22)$$

The first five G_n solutions have been obtained numerically; their reproduction by straightforward numerical integration requires the values of $G_n(0)$ and $G_n'(0)$ listed in Table II. For graphic displace the distributions of $G_n(\eta)$ are shown in figure 5 while the distribution of wall enthalpy in the form $(1 - g_w)(1 - g_c)^{-1}$ as given by four and five terms in the series is shown in figure 6.

The quantity which is of greatest applied interest from the energy solution obtained here is the distribution of wall enthalpy in terms of $g_w(\chi)$ obtainable from figure 6. Note that increasing $|\chi|$ corresponds to increasing x and that as x increases g_w changes from unity to g_c in the manner shown in figure 6. Again such behavior is symptomatic of nonsimilar flows.

If there is considered the comparative efficacy of injection on wedges and cones in making g_w approach g_c, then on the same basis as used above for the comparison of skin friction, it is found that injection is more effective on wedges than on cones.

In closing this section several remarks are perhaps in order. The solutions for both the velocity and energy fields have been obtained in terms of the transformed variables χ and η; the relation between the physical, streamwise coordinate x and χ is simply given in terms of the injection rate and of the external flow and cone characteristics. The transformation inverse to that of equation (1) yields $y = y(\chi, \eta)$ in terms of an integral with respect to η of the density ratio ρ_e/ρ and requires explicit determination of that density ratio. This in turn requires simultaneous consideration of the velocity, energy, and species fields and involves an equation of state, and relations between static temperature and static enthalpy of each species present. Calculations of this sort are straightforward and are of interest in connection with comparisons between experiment and analysis. Note that the applicability of the present solution for the energy field to a species field is discussed below.

CONCLUDING REMARKS

In reference 1 it is shown that the solutions for species and energy conservation in the two-dimensional case are identical provided no chemical reaction takes place, and provided a species mass balance across the porous surface is asserted. The same considerations apply to the cone case as well. The mathematical formulation for the species field involves replacing g in equation (17) by the species mass fraction, Y_i, replacing g_w, g_c and $(g_\eta)_w$ in the energy balance of equation (18) by $Y_{i,w}$, $Y_{i,e}$ and $[(Y_i)_\eta]_w$, respectively, and replacing equation (19) by $Y_i(\chi, \infty) = Y_i(0, \eta) = Y_{i,e}$. Then the solution for Y_i is sought in the form

$$Y_i(\chi, \eta) = Y_{i,e} + (Y_{i,c} - Y_{i,e}) \sum_{n=1}^{\infty} \chi^n G_n(\eta)$$

as in equation (20). Thus the solutions for $G_n(\eta)$ may be applied in determining the species fields as well.

It is also noted in reference 1 and repeated here for completeness that the analysis of species conservation is applicable to boundary layers with chemical reaction provided the species conservation is replaced by element conservation; in this case the species and temperature fields must be found from the solutions for element and energy conservation obtained here and from additional assumptions regarding the chemical behavior of the system, i. e., whether it corresponds to equilibrium chemistry or to finite rate behavior, and regarding the relations between static temperature and species enthalpy.

ACKNOWLEDGMENT

The author is pleased to acknowledge that Mrs. Judith Hays of the UCSD Computer Center performed the numerical analysis required here.

Table I. Initial values of the $N_n(\eta)$ functions.

n	$N''_n(0)$
1	0.9039
2	0.2504
3	-0.1068
4	0.04038
5	-0.07963

Table II. Initial values of the $G_n(\eta)$ functions.

n	$G'_n(0)$	$G_n(0)$
1	1.333	-2.465
2	3.287	-1.645
3	2.193	-0.2046
4	0.2728	-0.1205
5	0.1607	-0.1214

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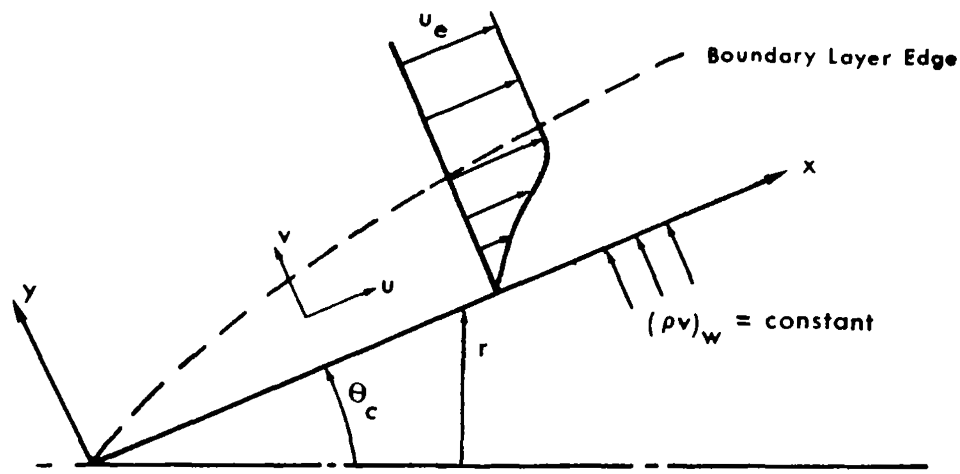

Figure 1. Schematic representation of the flow.

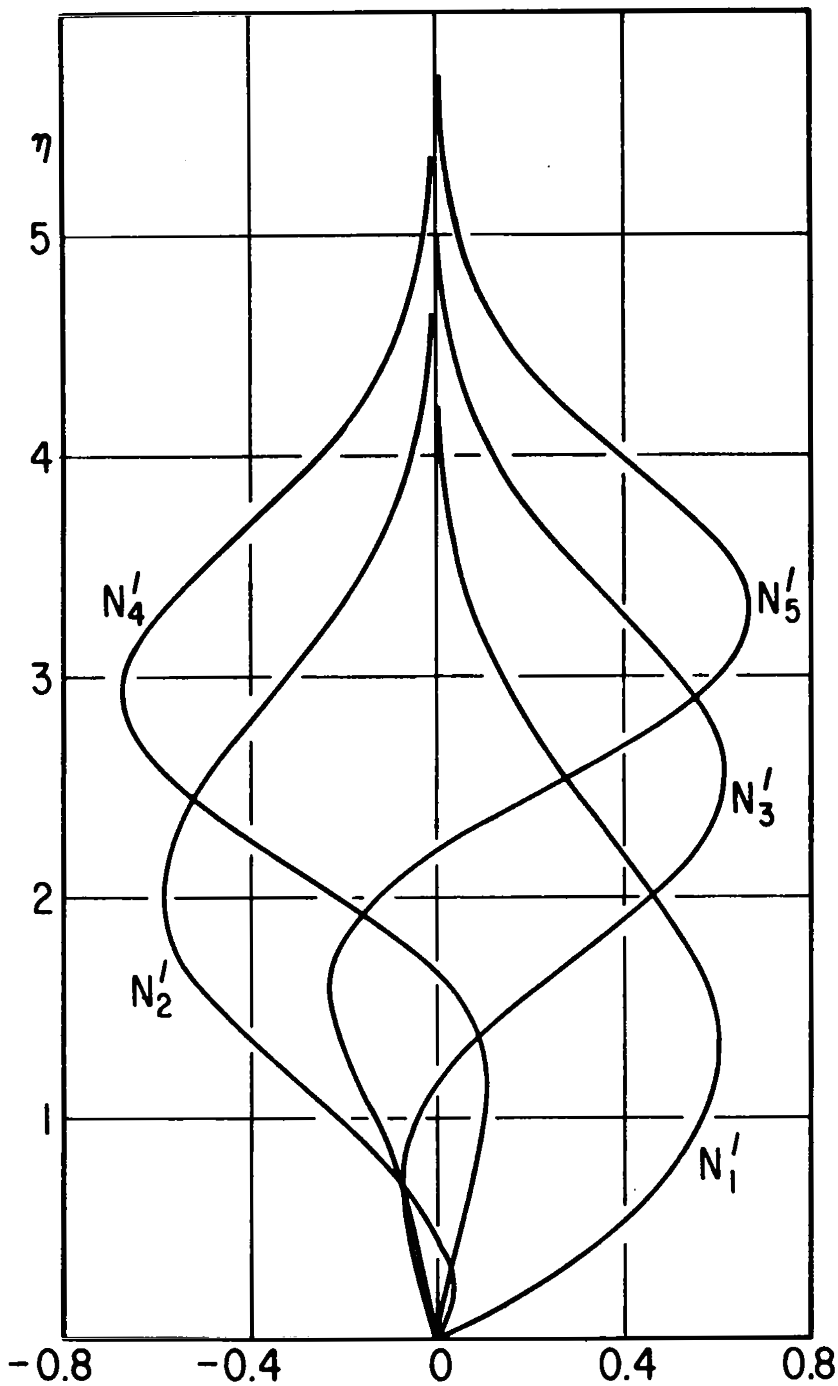

Figure 2. The functions $N'_n(\eta)$.

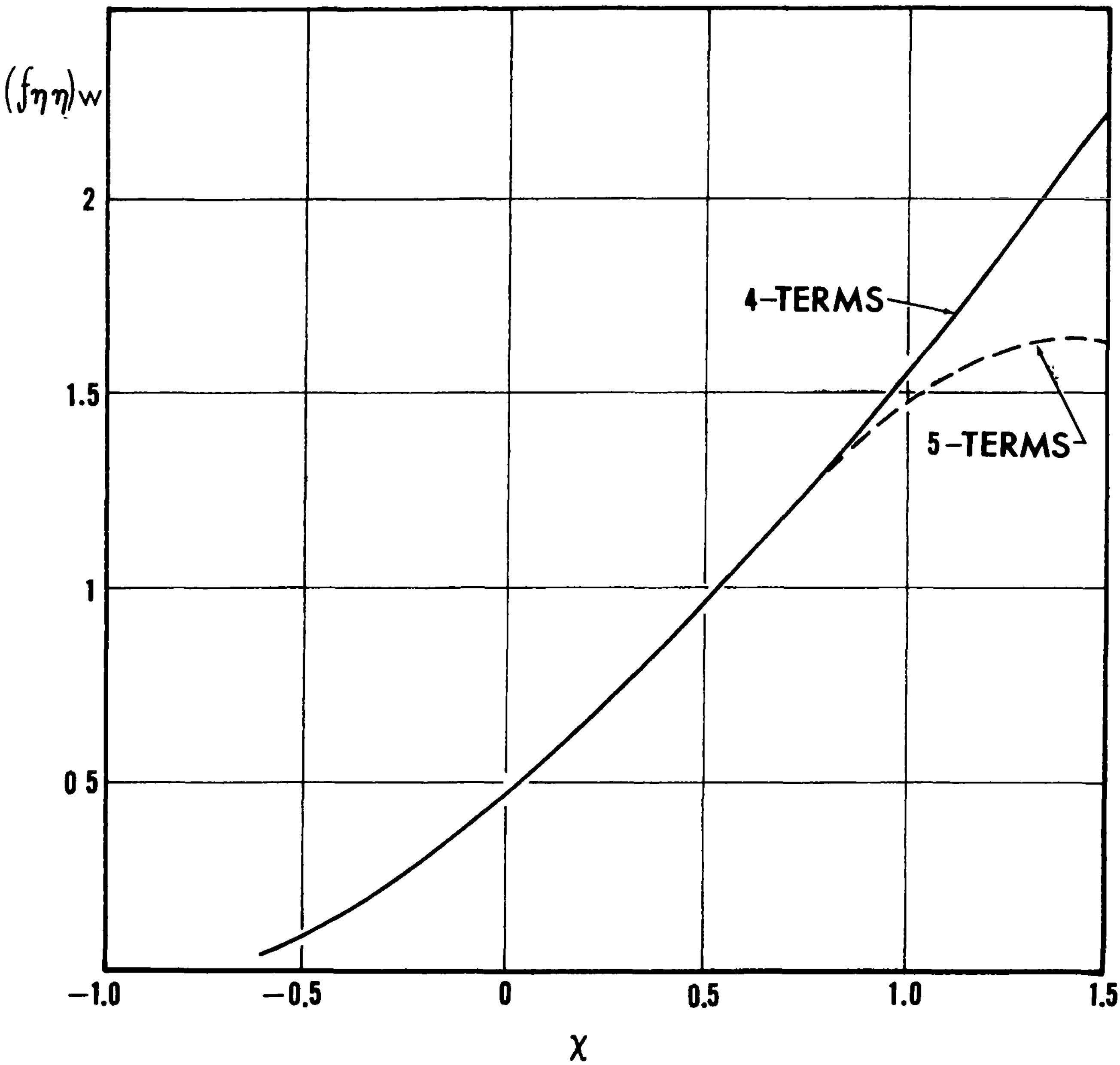

Figure 3. The effect of mass transfer on the skin-friction parameter.

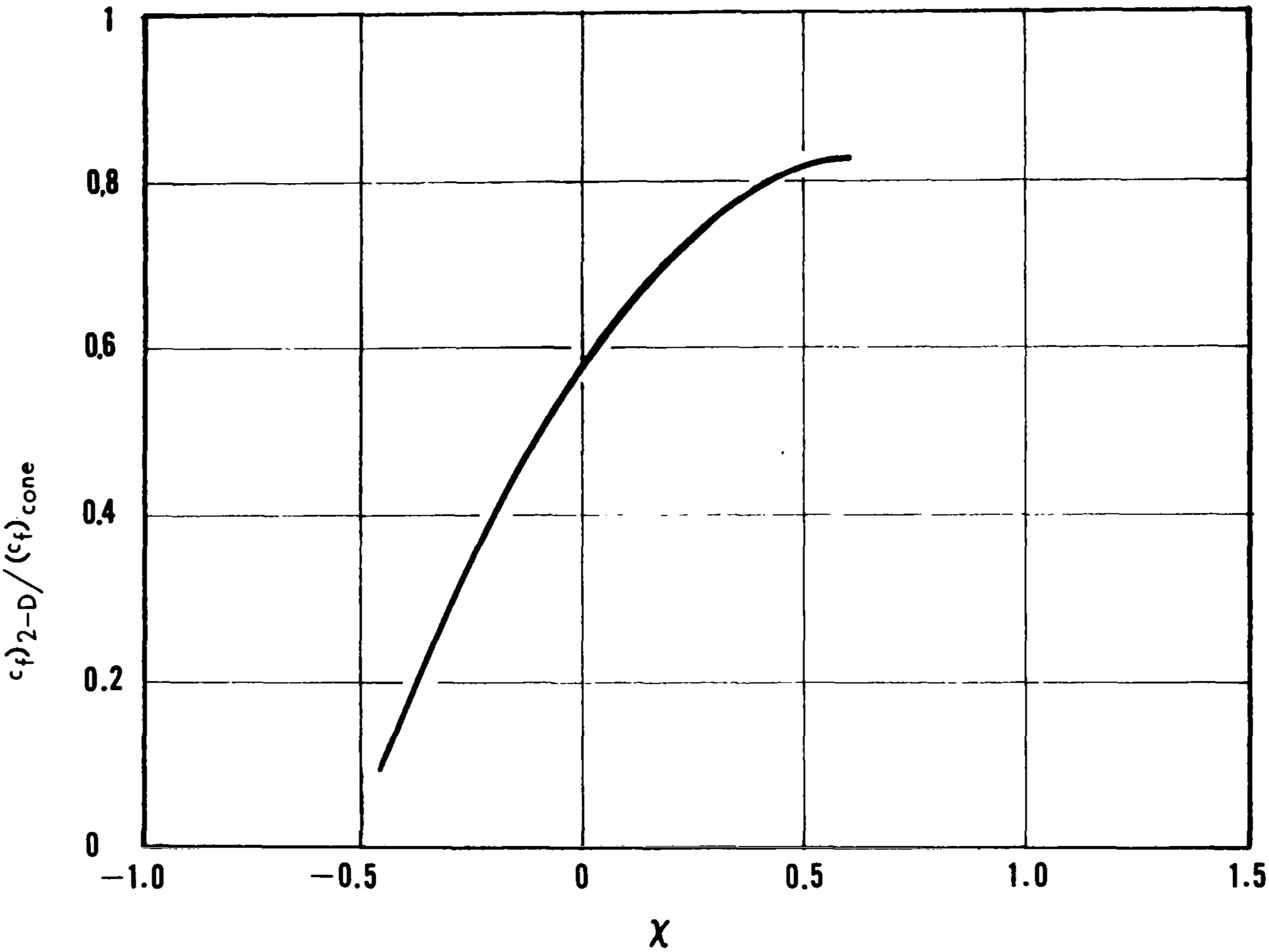

Figure 4. The relative effect of mass transfer on skin friction.

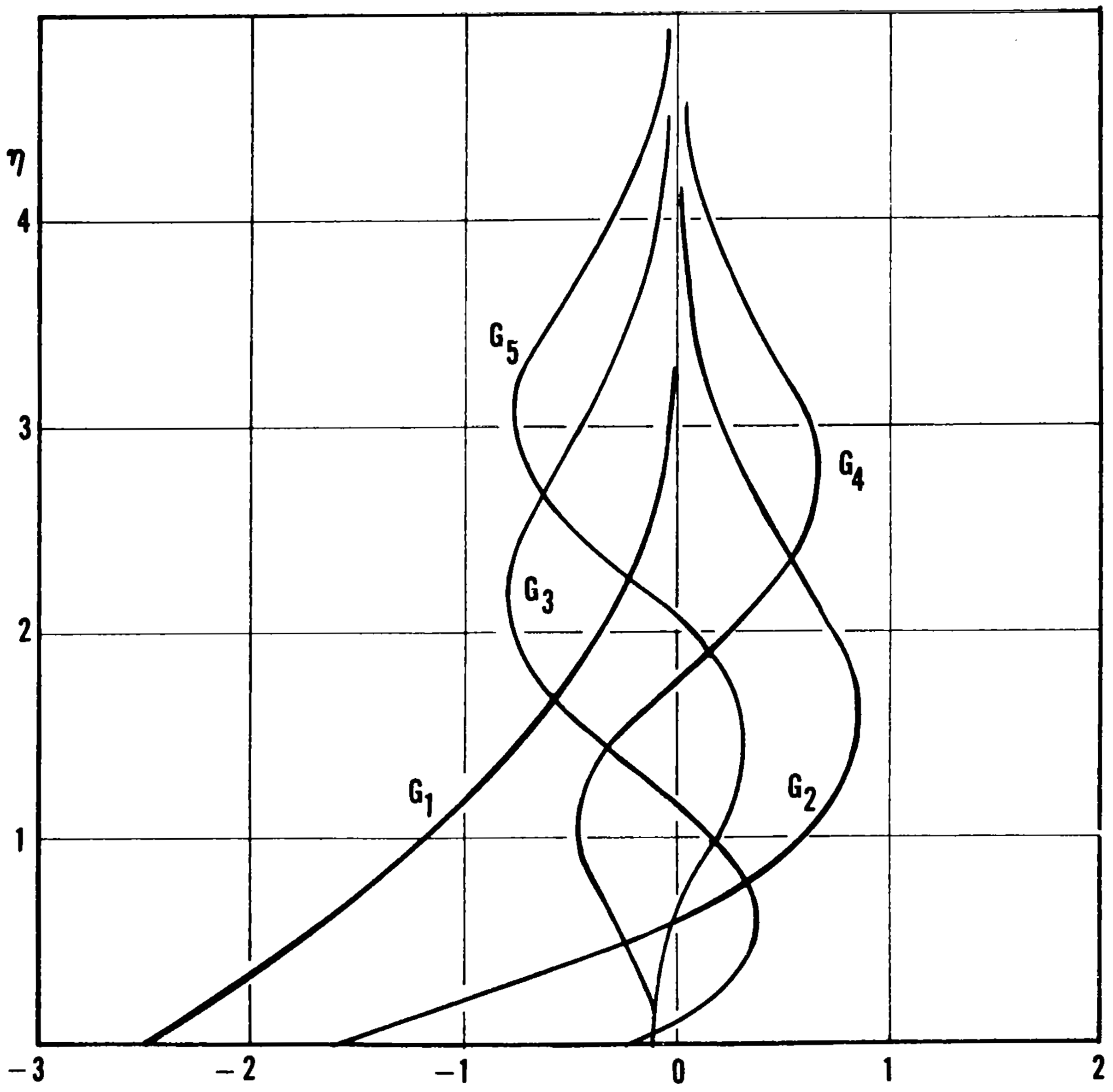

Figure 5. The functions $G_n(\eta)$.

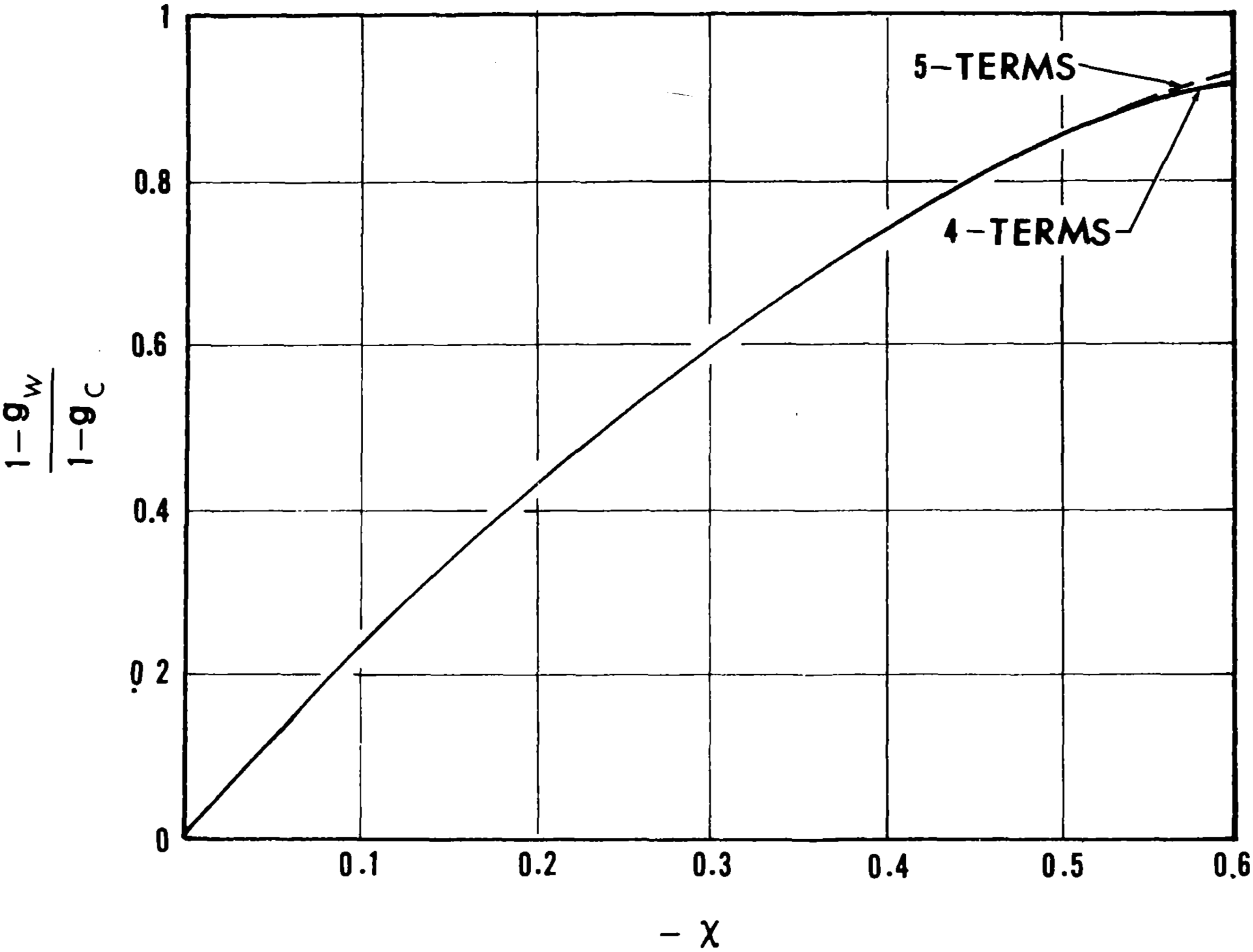

Figure 6. The variation of wall enthalpy with the injection parameter.